Asian Journal of Statistical Sciences
Vol. 4, No-2, (2024), pp. 141-154

ARF INDIA
Academic Open Access Publishing
Journal homepage link:
http://arfjournals.com/ajss

A new class of bimodal symmetric distributions

C. Satheesh Kumar[a] and Arathi S.[b]

[a] *Department of Statistics, University of Kerala, Trivandrum* [b] *Department of Statistics, University of Kerala, Trivandrum*

ARTICLE HISTORY
Compiled April 2, 2025

Received 14 February 2024; Accepted 20 September 2024

ABSTRACT
In this article, we consider a new class of bimodal symmetric distributions and study some of its important statistical properties. The estimation of parameter is attempted and illustrated with the help of certain real life data sets. A simulation study is carried out to examine the performance of the estimator of parameter of the distribution.

KEYWORDS
Bimodal distribution; Maximum likelihood estimation; Model selection; Simulation

1. Introduction

Bimodal distributions can arise in a wide variety of fields and has applications across different domains. Some important scenarios where bimodality arises include the size of worker ants in weaver colonies (Weber (1946)), the duration of volcanic eruptions (Azzalini and Bowman (1990)), amount of urine mercury in micromercurialism (Ely et al. (1999)), grain size distribution of sintered zirconia (Dierickx et al. (2000)), amount of tropospheric water vapour in tropics (Zhang et al. (2003)), gene expression pattern in breast cancer (Wang et al. (2009) and Ertel (2010)).

There is extensive research on modeling bimodality. Eisenberger(1964) investigated the conditions for which the density function of a mixture of two normal distributions. Behboodian (1970) shown that a mixture of two normal distributions is either unimodal or bimodal. Chosh (1978) provided a characterization of a bimodal probability distribution, contributing to the understanding of such models. Many authors have proposed different versions of the bimodal normal distribution to replace mixture distributions like Rao et al. (1988) and Sarma et al.(1990). These distributions did not materialize in the real world of statistics because of its functional form complexities. These models suffers serious estimation problems either from classical or Bayesian approaches as studied by McLachlan and Peel (2000). Two-component mixture distributions are often used as a powerful tool for modeling bimodal data. A major issue with these distributions is that it is necessary to deal with problem of non-identifiability of their parameters, see McLachlan et al. (2019) for more details.

CONTACT C. Satheesh Kumar[a]. Email:drcsatheeshkumar@gmail.com

Through this article we introduce a new class of bimodal distributions in an infinite domain which is symmetric in nature. The proposed distribution is named as "bimodal symmetric distribution (BSD)". We study some important statistical properties of the BSD. The paper is organized as follows: In Section 2, we present the definition of the model followed by some important results and derived its cumulative distribution function, moments, generating functions, quantiles, entropy and reliability measures. The estimation of parameter of the model is discussed in Section 3. The estimation procedure is illustrated using some real life data sets in Section 4 to highlight the usefulness of the model. A brief simulation study is conducted in Section 5 to analyse the performance of the maximum likelihood estimator of parameter of the distribution followed by conclusion in Section 6.

2. Definition and Properties

In this section, we present the definition and some important properties of bimodal symmetric distribution.

Definition 2.1. A continuous random variable X is said to follow bimodal symmetric distribution (BSD) if its p.d.f is of the form

$$f(x;\beta) = \beta e^{-\beta|x|}\left(1 - e^{-\beta|x|}\right); -\infty < x < \infty \tag{1}$$

for $\beta > 0$.

Here β is the scale parameter. A distribution with p.d.f (1) hereafter is denoted as BSD(β).
When β=1, the p.d.f (1) of BSD(β) reduces to

$$f_1(x) = e^{-|x|}\left(1 - e^{-|x|}\right); -\infty < x < \infty. \tag{2}$$

Proposition 2.2. *A BSD(β) has two modes.*

Proof. On differentiating (1) with respect to x, we have

$$\frac{d}{dx}f(x;\beta) = f'(x;\beta) = \begin{cases} \beta^2 e^{\beta x}(1 - 2e^{\beta x}) & ; x < 0 \\ \beta^2 e^{-\beta x}(2e^{-\beta x} - 1); & x \geq 0 \end{cases} \tag{3}$$

On equating equation (3) to 0, the zero of the first expression happens at $x_0 = \frac{1}{\beta}\log\left(\frac{1}{2}\right)$ and that of the second expression at $x_0 = \frac{1}{\beta}\log(2)$. Therefore, we can say that the BSLD is bimodal in nature irrespective of the values of parameter β. □

The probability plots of the BSD(β) are presented in the Figure 1 for particular values of β. The bimodal nature of BSD can be clearly observed from Figure 1.

Proposition 2.3. *If X has BSD(β), then Z=-X also has BSD(β).*

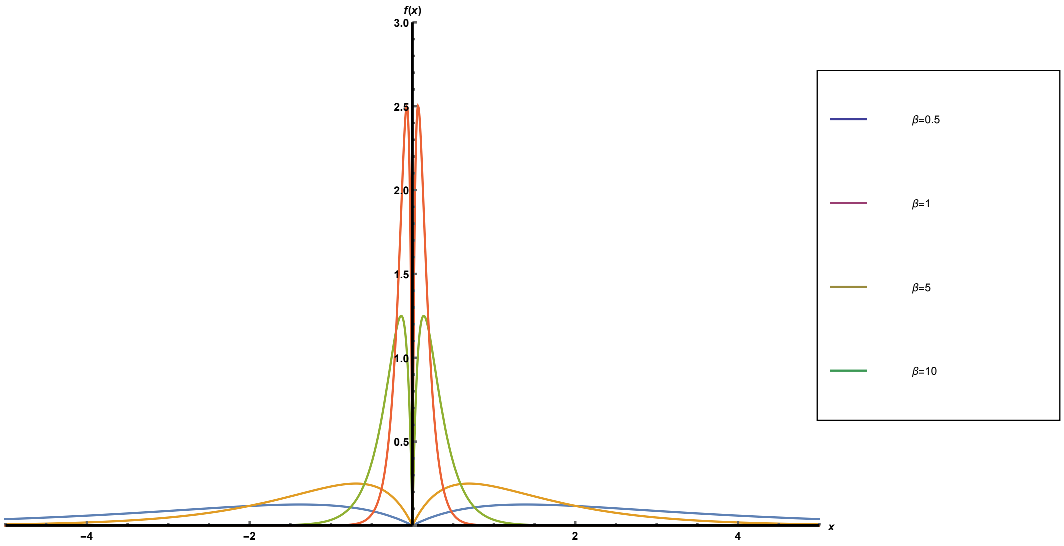

Figure 1. Illustrations of BSD(β) for various choices of β.

Proof. For any z$\in \mathbb{R}$ and $\beta > 0$, the p.d.f of random variable Z is as follows

$$\begin{aligned} f_1(z) &= f(-z,\beta)\left|\frac{dx}{dz}\right| \\ &= f(z,\beta). \end{aligned}$$

□

Proposition 2.4. *If X has BSD(β), then Z=$|X|$ follows generalized exponential distribution (GE(2,β)) of Gupta and Kundu (2001) with p.d.f given by*

$$f_2(z) = 2\beta e^{-\beta z}\left(1 - e^{-\beta z}\right) \ ; \ z > 0. \tag{4}$$

Proof. For $z > 0$, the p.d.f of Z=$|X|$ is as follows

$$\begin{aligned} f_2(z) &= f(-z;\beta)\left|\frac{dx}{dz}\right| + f(z;\beta)\left|\frac{dx}{dz}\right| \\ &= f(z;\beta) + f(z;\beta) \end{aligned}$$

which leads to equation(13). □

Proposition 2.5. *If X has BSD(β), then p.d.f of Z=X^2 is the following*

$$f_3(z) = \frac{\beta}{\sqrt{z}} e^{-\beta\sqrt{z}}\left(1 - e^{-\beta\sqrt{z}}\right) \ ; \ z > 0. \tag{5}$$

Proof. For $z > 0$, the p.d.f of Z=X^2 can be obtained as

$$\begin{aligned} f_3(z) &= f\left(-\sqrt{z};\beta\right)\left|\frac{dx}{dz}\right| + f\left(\sqrt{z};\beta\right)\left|\frac{dx}{dz}\right| \\ &= f\left(\sqrt{z};\beta\right) + f\left(\sqrt{z};\beta\right) \end{aligned}$$

which on simplifying leads to equation(12). □

- It is to be noted that sum and difference of bimodal symmetric random variables have the same distribution.

Proposition 2.6. *The cumulative distribution function (c.d.f) of BSD(β) is given by*

$$F(x;\beta) = \begin{cases} e^{\beta x}\left(1 - \frac{e^{\beta x}}{2}\right); x < 0 \\ 1 - e^{-\beta x}\left(1 - \frac{e^{-\beta x}}{2}\right); x \geq 0. \end{cases} \tag{6}$$

Proof. For $x > 0$, the c.d.f is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \beta e^{\beta x}\left(1 - e^{\beta x}\right) dx \\ &= \beta\left[\frac{e^{\beta x}}{\beta} - \frac{e^{2\beta x}}{2\beta}\right]_{-\infty}^{x} \\ &= \frac{e^{\beta x}}{2}\left[2 - e^{\beta x}\right] \end{aligned}$$

and the c.d.f for $x \geq 0$ is

$$\begin{aligned} F(x) &= \int_{-\infty}^{0} \beta e^{\beta x}\left(1 - e^{\beta x}\right) dx + \int_{0}^{x} \beta e^{-\beta x}\left(1 - e^{-\beta x}\right) dx \\ &= \beta\left[\frac{e^{\beta x}}{\beta} - \frac{e^{2\beta x}}{2\beta}\right]_{-\infty}^{0} + \beta\left[\frac{e^{-\beta x}}{-\beta} - \frac{e^{2\beta x}}{-2\beta}\right]_{0}^{x} \\ &= \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - e^{-\beta x}\left(1 - \frac{e^{-\beta x}}{2}\right)\right] \\ &= 1 - \frac{1}{2}e^{-\beta x}\left(2 - e^{-\beta x}\right). \end{aligned}$$

□

Proposition 2.7. *For $r \geq 1$, the r^{th} raw moment μ_r' of the BSD(β) with p.d.f (1) is the following*

$$\mu_r' = \frac{r!}{\beta^r}\left(1 + (-1)^r\right)\left(1 - \frac{1}{2^{r+1}}\right). \tag{7}$$

Proof. By definition, the r^{th} raw moment of TPSLD is obtained as

$$\begin{aligned}
\mu'_r &= E(X^r) \\
&= \beta\left(\int_{-\infty}^{0} x^r e^{\beta x}(1-e^{\beta x})dx + \int_{0}^{\infty} x^r e^{-\beta x}(1-e^{-\beta x})dx\right) \\
&= \beta\left(\frac{(-1)^r\Gamma(r+1)}{\beta^{r+1}} - \frac{(-1)^r\Gamma(r+1)}{(2\beta)^{r+1}} + \frac{\Gamma(r+1)}{\beta^{r+1}} - \frac{\Gamma(r+1)}{(2\beta)^{r+1}}\right) \\
&= \frac{\beta\Gamma(r+1)}{\beta^{r+1}}\left[(-1)^r\left[1-\frac{1}{2^{r+1}}\right] + \left[1-\frac{1}{2^{r+1}}\right]\right] \\
&= \frac{r!}{\beta^r}\left[((-1)^r+1)\left(1-\frac{1}{2^{r+1}}\right)\right].
\end{aligned}$$

□

As a consequence of **Result 2.7**, we have the following remarks.

Remark 1. The mean and variance of BSD(β) are as follows

$$E(X) = \mu'_1 = 0 \tag{8}$$

and

$$\mu_2 = Variance = \frac{7}{2\beta^2}. \tag{9}$$

Remark 2. The BSD(β) is overdispersed for any $\beta > 0$.

Remark 3. The measure of skewness and kurtosis of the $BSD(\beta)$ are given by

$$\beta_1 = 0 \tag{10}$$

and

$$\beta_2 = 3.79. \tag{11}$$

It should be noted that BSD is always symmetric and leptokurtic in nature.

Proposition 2.8. *For any $t \in \mathbb{R}$ and $i = \sqrt{-1}$, the characteristic function of the BSD(β) is given by*

$$\Phi_X(t) = \frac{4\beta^4 - 2\beta^2 t^2}{(it+\beta)(it+2\beta)(it-\beta)(it-2\beta)}. \tag{12}$$

Proof. By definition, characteristic function of BSD is

$$\begin{aligned}
\Phi_X(t) &= E\left(e^{itX}\right) \\
&= \beta\left(\int_{-\infty}^{0} e^{itx}\left(e^{\beta x} - e^{2\beta x}\right)dx + \int_{0}^{\infty} e^{itx}\left(e^{-\beta x} - e^{-2\beta x}\right)dx\right) \qquad (13)
\end{aligned}$$

On integrating (13), w get

$$\begin{aligned}\Phi_X(t) &= \beta\left(\frac{1}{(it+\beta)} - \frac{1}{(it+2\beta)} - \frac{1}{(it-\beta)} + \frac{1}{(it-2\beta)}\right)\\ &= 2\beta^2\left(\frac{2\beta^2-t^2}{(it+\beta)(it+2\beta)(it-\beta)(it-2\beta)}\right)\end{aligned}$$

which leads to (12). □

The quantile function is a powerful tool that allows you to find the value of a random variable corresponding to a given probability, and it is commonly used in statistical analysis, simulations, and risk assessments. It is the inverse of the cumulative distribution function (CDF). It plays a significant role in statistical analysis, as it allows analysts to interpret data and distributions in terms of specific percentiles or probabilities. The quantile function of BSD is obtained as follows.

Proposition 2.9. *The Quantile function of BSD(β) is given by*

$$Q(t) = X_t = \begin{cases} \frac{1}{\beta}\log\left(1-\sqrt{(1-2t)}\right); 0 < t \le \frac{1}{2} \\ \frac{1}{\beta}\log\left(1-\sqrt{(2t-1)}\right); \frac{1}{2} \le t < 1. \end{cases} \tag{14}$$

Proof. For $0 < t \le \frac{1}{2}$, the quantile function of BSD is

$$\begin{aligned} F(X_t) &= t\\ e^{\beta x_t}\left(1-\frac{e^{\beta x_t}}{2}\right) &= t\\ \Rightarrow 2e^{\beta x_t} - e^{2\beta x_t} &= 2t\\ \Rightarrow (1-e^{\beta x_t})^2 &= 1-2t\\ \Rightarrow e^{\beta x_t} &= 1-\sqrt{(1-2t)}\\ \Rightarrow x_t &= \frac{1}{\beta}\log\left(1-\sqrt{(1-2t)}\right)\end{aligned}$$

and the quantile function of BSD for $\frac{1}{2} \le t < 1$ is

$$\begin{aligned} 1-e^{-\beta x_t}\left(1-\frac{e^{-\beta x_t}}{2}\right) &= t\\ \Rightarrow 2e^{-\beta x_t} - e^{-2\beta x_t} &= 2(1-t)\\ \Rightarrow (1-e^{-\beta x_t})^2 &= 2t-1\\ \Rightarrow e^{-\beta x_t} &= 1-\sqrt{(2t-1)}\\ \Rightarrow x_t &= \frac{1}{\beta}\log\left(1-\sqrt{(2t-1)}\right).\end{aligned}$$

□

Rényi entropy is a generalization of the Shannon entropy, a fundamental concept in information theory. While Shannon entropy measures the uncertainty or randomness of a probability distribution, Rényi entropy introduces a parameter γ that allows for the tuning of sensitivity to different probabilities. This gives it a broader range of applications in various fields, including information theory, cryptography, machine learning, and statistical analysis.

Proposition 2.10. *The Rényi entropy of the $BSD(\beta)$ is given by*

$$I_R(\gamma) = \frac{1}{\gamma - 1} \log\left(2\beta^{\gamma-1} B_1(\gamma, \gamma+1)\right) \tag{15}$$

where $\gamma > 0$, $\gamma \neq 1$ and $B_r(a,b) = \int_0^r t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete beta function.

Proof. By definition, Rényi entropy of a distribution with p.d.f (.) is

$$I_R(\gamma) = \frac{1}{\gamma - 1} \log \int_{-\infty}^{\infty} g^{\gamma}(x)dx$$

where $\gamma > 0$ and $\gamma \neq 1$. Thus, the Rényi entropy of BSD with p.d.f (1) is

$$I_R(\gamma) = \frac{1}{\gamma - 1} \log \int_{-\infty}^{\infty} f^{\gamma}(x)dx \tag{16}$$

in which

$$\begin{aligned} \int_{-\infty}^{\infty} f^{\gamma}(x)\,dx &= \int_{-\infty}^{0} f^{\gamma}(x)\,dx + \int_{0}^{\infty} f^{\gamma}(x)\,dx \\ &= \beta^{\gamma}\left(\int_{-\infty}^{0} e^{\gamma\beta x}\left(1-e^{\beta x}\right)^{\gamma} dx + \int_{0}^{\infty} e^{-\gamma\beta x}\left(1-e^{-\beta x}\right)^{\gamma} dx\right) \end{aligned} \tag{17}$$

Put $e^{\beta x} = v$ and $e^{-\beta x} = u$ in equation (17) to get

$$\begin{aligned} \int_{-\infty}^{\infty} f^{\gamma}(x)\,dx &= \beta^{\gamma}\left(\int_{0}^{1} \frac{v^{\gamma}(1-v)^{\gamma}}{\beta v} dv + \int_{0}^{1} \frac{u^{\gamma}(1-u)^{\gamma}}{\beta u} du\right) \\ &= \beta^{\gamma}\left(\frac{1}{\beta} + \frac{1}{\beta}\right) B_1(\gamma, \gamma+1) \end{aligned} \tag{18}$$

Substituting (18) in equation (16) leads to **Result 2.10**. □

Reliability measures are important tools used in various fields like engineering, manufacturing, healthcare etc. to assess the performance, longevity, and dependability of systems or components over time. These measures help in predicting the likelihood of failure, estimating the expected lifespan, and optimizing maintenance schedules. Some key reliability measures like survival function, hazard rate function, reversed hazard rate function and mean residual life function are presented below.

Proposition 2.11. *The survival function of the BSD(β) is given by*

$$S(x) = \begin{cases} 1 - \frac{1}{2}e^{\beta x}\left(2 - e^{\beta x}\right); x < 0 \\ \frac{1}{2}e^{-\beta x}\left(2 - e^{-\beta x}\right); x \geq 0 \end{cases} \tag{19}$$

The proof immedietly follows from the definition of survival function $S(x) = 1 - F(x)$ and hence omitted.

Proposition 2.12. *The hazard rate function (or failure rate function) of the BSD(β) is as follows*

$$h(x) = \begin{cases} \frac{2\beta e^{\beta x}(1-e^{\beta x})}{2-\beta e^{\beta x}(1-e^{\beta x})}; x < 0 \\ \frac{2\beta(1-e^{-\beta x})}{(2-e^{-\beta x})} \quad ; x \geq 0. \end{cases} \tag{20}$$

The proof directly follows from the definition of hazard rate function and therefore omitted. The failure rate function of BSD is graphically represented in Figure 3.

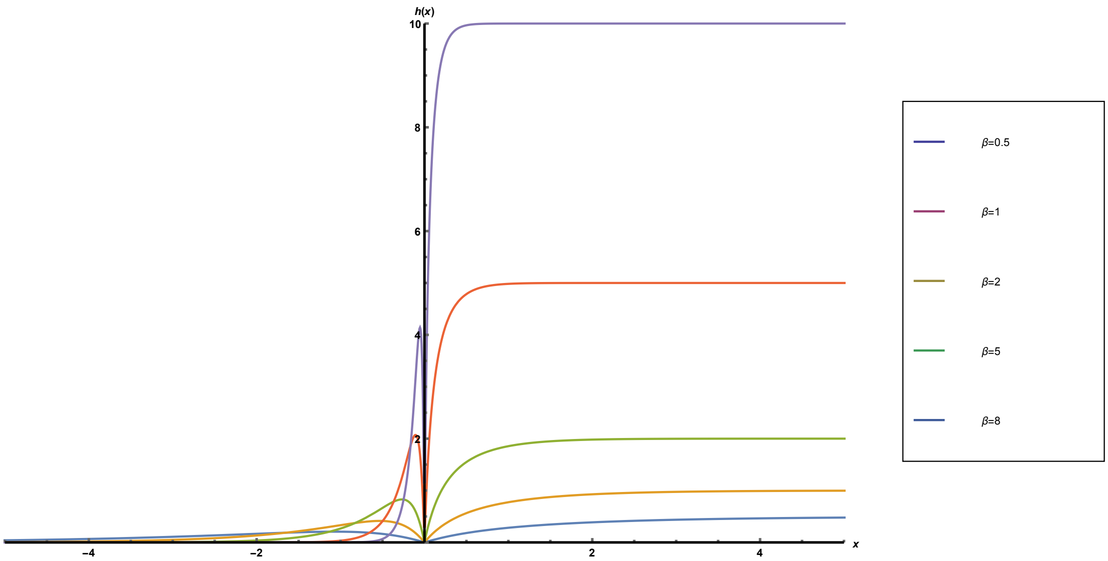

Figure 2. Failure rate plot of BSD(β) for various choices of β.

Proposition 2.13. *The reversed hazard rate function of the BSD(β) is as follows*

$$r(x) = \begin{cases} \frac{2\beta(1-e^{\beta x})}{(2-e^{\beta x})}; x < 0 \\ \frac{2\beta e^{-\beta x}(1-e^{-\beta x})}{2-e^{-\beta x}(2-e^{-\beta x})}; x \geq 0. \end{cases} \tag{21}$$

The proof immediately follows the definition of reversed hazard rate function $r(x) = \frac{f(x)}{F(x)}$ and hence omitted.

Proposition 2.14. *The mean residual life function (MRLF) of the BSD(β) is given* *by,*

$$m(x) = \begin{cases} \frac{-3-4\beta x+4e^{\beta x}-e^{2\beta x}}{2\beta(2-2e^{\beta x}+e^{2\beta x})}; x < 0 \\ \frac{4-e^{-\beta x}}{2\beta(2-e^{-\beta x})}; x \geq 0. \end{cases} \tag{22}$$

Proof. By defintion, the mean residual life function (MRLF) is given by

$$m(x) = E[X - x|X > x].$$

For $x < 0$, the MRLF of BSD with p.d.f (1) is

$$\begin{aligned} m(x) &= \frac{1}{1-F(x)} \int_x^0 (1 - F(t))dt \\ &= \frac{1}{1-e^{\beta x}\left(1-\frac{e^{\beta x}}{2}\right)} \int_x^0 1 - e^{\beta t}\left(1-\frac{e^{\beta t}}{2}\right) dt \\ &= \frac{1}{2-2e^{\beta x}+e^{2\beta x}} \int_x^0 (2 - 2e^{\beta t} + e^{2\beta t})dt \\ &= \frac{-3-4\beta x+4e^{\beta x}-e^{2\beta x}}{2\beta(2-2e^{\beta x}+e^{2\beta x})} \end{aligned}$$

and for $x \geq 0$, the MRLF of BSD is

$$\begin{aligned} m(x) &= \frac{1}{1-F(x)} \int_x^\infty (1 - F(t))dt \\ &= \frac{1}{e^{-\beta x}\left(1-\frac{e^{-\beta x}}{2}\right)} \int_x^\infty e^{-\beta t}\left(1-\frac{e^{-\beta t}}{2}\right) dt \\ &= \frac{1}{2e^{-\beta x}+e^{-2\beta x}} \int_x^\infty (2e^{-\beta t} + e^{-2\beta t})dt \\ &= \frac{4-e^{-\beta x}}{2\beta(2-e^{-\beta x})}. \end{aligned}$$

□

3. Estimation

In this section, we discuss two estimation procedures for estimating the parameter of bimodal symmetric distribution - the method of moment estimation and the method of maximum likelihood estimation.

3.1. Method of Moments

The method of moments is relatively a simple procedure to estimate the unknown parameters of probability distributions. The moment estimator of the parameter β of BSD can be obtained by equating the second theoretical moment with the corresponding sample moment. The moment estimator of the parameter β of BSD is given by

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= \frac{7}{2\beta^2} \\ \Rightarrow \hat{\beta}_{MM} &= \sqrt{\frac{7n}{2\sum\limits_{i=1}^{n} X_i^2}} \end{aligned} \tag{23}$$

since $\beta > 0$.

3.2. Maximum Likelihood Estimation

In this section, we consider the estimation of the parameter β of BSD using the maximum likelihood estimation procedure.

Let $X_1, X_2, X_3, ..., X_n$ be a random sample of size n from BSD with pdf (1) . Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding ordered sample. Then, the log-likelihood function $l(\beta)$ of the sample is given by

$$l(\beta) = n\log\beta + \beta\sum_{I_1} X_i + \log\left(1 - e^{\beta\sum\limits_{I_1} X_i}\right) - \beta\sum_{I_2} X_i + \log\left(1 - e^{-\beta\sum\limits_{I_2} X_i}\right) \tag{24}$$

where $\sum\limits_{I_i}$ denotes the summation over the set I_i such that $I_1 = \{i : Y_i < 0,\ for\ i = 1, 2, ..., s\}$ and $I_2 = \{i : Y_i \geq 0,\ for\ i = s+1, ..., n\}$.

Assume that $\hat{\beta}$ be the maximum likelihood estimator of the parameter β of BSD. On differentiating the log-likelihood function given in equation (24) with respect to the parameter β and equating to zero, we obtain the following likelihood equation.

$$\frac{\partial l(\beta)}{\partial\beta} = 0$$

or equivalently,

$$\sum_{I_1} X_i - \frac{\sum\limits_{I_1} X_i e^{\beta\sum\limits_{I_1} X_i}}{\left(1 - e^{\beta\sum\limits_{I_1} X_i}\right)} - \sum_{I_2} X_i + \frac{\sum\limits_{I_2} X_i e^{-\beta\sum\limits_{I_2} X_i}}{\left(1 - e^{-\beta\sum\limits_{I_2} X_i}\right)} = 0 \tag{25}$$

On solving we get

$$\left(\sum_{I_1} X_i - \sum_{I_2} X_i\right)\left(1 - 2e^{\beta\sum\limits_{I_1} X_i - \beta\sum\limits_{I_2} X_i}\right) - \left(\sum_{I_1} X_i e^{-\beta\sum\limits_{I_2} X_i} - \sum_{I_2} X_i e^{\beta\sum\limits_{I_1} X_i}\right)$$
$$-2\left(\sum_{I_1} X_i e^{\beta\sum\limits_{I_1} X_i} - \sum_{I_2} X_i e^{-\beta\sum\limits_{I_2} X_i}\right) = 0 \quad (26)$$

On solving the equation (26) we can obtain the maximum likelihood estimator of the parameter β of BSD.

4. Applications

In this section, we illustrate the usefulness of the proposed model BSD(β) using four real life data sets. The model is compared with bimodal-symmetric Normal distribution (BND) of Elal Olivero (2010), bimodal Cauchy distribution (BCD) of Hassan and Hijazi (2010) and bimodal-symmetric Laplace distribution (BLD) of Harandi and Alamatsaz (2013). Firstly, we use National Consumer Price Index (INPC) data to highlight the suitability of the BSD(β). A data set of 13 observations which denotes percentage change in National Consumer Price Index (INPC) for the wearing apparel of different areas of Brazil. The data is collected in the period of August 1 to August 29, 2017 (reference period) and is compared with prices charged in the period of June 29 to July 31, 2017 (base period). The data is taken from *www.ibge.gov.br*. The second data set is taken from World Population Prospects: 2019 revision which can be viewed at *https://population.un.org/wpp/downloads*. The data set is on the average annual rate of population change (expressed in percentage) of Russian Federation and has 30 observations. The third data set appears in macro-economic framework statement 2018-2019. It has 15 observations representing percentage change of government finances (expressed in crores) for 2016-2017 in macro-economic framework statement. The fourth data set is on real annual returns on stocks from Benderly and Zwick Data available in Benderly and Zwick (1985). It has 28 observations.

Data set 1: 0.53, -0.68, 0.68, 0.73, 0.26, 0.88, 0.14, 0.56, 0.90, -0.25, 0.18, 0.10, -0.54.
Data set 2: -0.060, -0.027, -0.019, -0.054, -0.123, -0.202, -0.258, -0.270, -0.248, -0.219, -0.216,
-0.251, -0.300, -0.311, -0.248, -0.110, 0.131, 0.209, -0.027, -0.377, -0.247, 0.094, 0.630, 0.698, 0.625, 0.554, 0.568, 1.077, 1.474, 1.599.
Data set 3: 24.8, 33.6, 1.0, -2.7, 14.7, 83.1, -5.3, 12.6, 12.6, 16.4, 5.6, -10.4, 0.8, -5.3, -17.2.
Data set 4: 53.0, 31.2, 3.7, -13.8, 41.7, 10.5, -1.3, 26.1, -10.5, 21.2, 15.5, 10.2, -13.3, 21.3, 6.8, -13.5, -0.4, 10.5, 15.4, -22.6, -37.3, 31.2, 19.1, -13.1, -1.3, 8.6, 22.2, -12.2.

We have numericaly illustrated the maximum likelihood estimation procedure using four real life data sets. We have fitted BSD to all the four data sets and considered the fitting of the models-BND, BCD and BLD for comparison using MATHEMATICA. We have computed the P-value and Kolmogorov Smirnov Statistic (KSS) value associated in all the cases. From Table 1, we can observe that BSD has the maximum P-value and

Table 1. Estimated values of the parameters for the model: $BND(\mu, \sigma)$, $BCD(\mu, \sigma)$, $BLD(\mu, \sigma)$ and $BSD(\beta)$ with respective log-likelihood value, KSS value, P-value, AIC, BIC and AICc values for all four data sets.

Dataset	Estimates of the parameters	$BND(\mu, \sigma)$	$BCD(\mu, \sigma)$	$BLD(\mu, \sigma)$	$BSD(\beta)$
1	$\hat{\beta}$	-	-	-	2.9915
	$\hat{\mu}$	1.3564	0.1175	-0.3256	-
	$\hat{\sigma}$	0.6901	0.3644	0.2272	-
	l	-17.5756	-72.8693	-14.8704	-10.7507
	KSS	0.5338	0.9853	0.4137	0.3027
	P-value	0.0005	1.7272x10^{-12}	0.0157	0.1492
	AIC	39.1512	149.7390	33.7408	23.5014
	BIC	40.2811	150.8680	34.8707	24.0064
	AICc	40.3512	150.9390	34.9408	23.8651
2	$\hat{\beta}$	-	-	-	4.1207
	$\hat{\mu}$	1.1502	-0.5024	1.1432	-
	$\hat{\sigma}$	0.6585	0.4612	0.3531	-
	l	-39.3858	-158.0510	-42.8160	-24.0551
	KSS	0.4633	0.9845	0.4798	0.2057
	P-value	2.0393x10^{-6}	4.7805x10^{-27}	7.1049x10^{-7}	0.1366
	AIC	82.7717	320.1020	89.6321	50.1101
	BIC	85.5741	322.904	92.4345	51.5113
	AICc	83.2161	320.546	90.0765	50.2530
3	$\hat{\beta}$	-	-	-	0.0949
	$\hat{\mu}$	22.0847	13.7735	20.6439	-
	$\hat{\sigma}$	14.9099	11.6589	6.7666	-
	l	-74.5309	-137.5090	-69.9821	-69.9945
	KSS	0.3819	0.9844	0.3544	0.2885
	P-value	0.0177	3.4481x10^{-14}	0.0343	0.1341
	AIC	153.0620	279.0180	143.9640	141.9890
	BIC	154.4780	280.4340	145.3800	142.6970
	AICc	154.0620	280.0180	144.9640	142.297
4	$\hat{\beta}$	-	-	-	0.08604
	$\hat{\mu}$	60.7130	2.2290	17.1121	-
	$\hat{\sigma}$	32.8421	13.5930	5.9074	-
	l	-143.9350	-243.1570	-132.0980	-127.1360
	KSS	0.5047	0.9846	0.2931	0.2086
	P-value	3.9451x10^{-7}	2.42578x10^{-25}	0.0126	0.1509
	AIC	291.8690	490.3140	268.1960	256.2730
	BIC	294.5340	492.9790	270.8610	257.6050
	AICc	292.3490	490.7940	268.6760	256.4270

least KSS value. For model comparison, we have used some well known informaton measures- Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and corrected Akaike's Information Criterion (AICc) and results are presented in Table 1. From Table 1, it can be viewed that BSD has the least AIC, BIC and AICc value as compared to other models thereby indicating the suitability of BSD in modelling these data sets. Based on the computed values, it can be viewed that BSD provides best fit compared to the existing models- BND, BCD and BLD. Hence, it can be viewed as an alternative to BND, BCD and BLD in several practical situations.

5. Simulation

In this section, we analyse the performance of the maximum likelihood estimator of the proposed model by conducting a simulation study. Using the software MATHEMATICA, a finite sample of size (n) is generated using inverse CDF transformation and the parameter is estimated using maximum likelihood procedure. We have generated

Table 2. The parameter estimates, their corresponding absolute bias and MSEs for the simulated samples.

Parameter	Sample size (n)	Estimates	MSE	Absolute bias
β=0.4	20	0.4240	0.0059	0.0240
	50	0.4113	0.0018	0.0113
	100	0.3979	0.00104	0.0021
	200	0.4013	0.0004	0.0013
	500	0.4007	0.0001	0.0007
β=1.0	20	1.0239	0.0272	0.0239
	50	1.0057	0.0109	0.0057
	100	0.9951	0.008093	0.0048
	200	1.0030	0.0017	0.0030
	500	0.9984	0.0011	0.0016
β=2.0	20	2.0336	0.1458	0.0336
	50	1.9871	0.0411	0.0129
	100	2.0114	0.0231	0.0114
	200	2.0093	0.0136	0.0093
	500	2.0076	0.0053	0.0076

60 independent samples of sizes ranging from 20 to 500 from BSD and computed their mean square errors and absolute bias. The initial values of the parameter β used to generate data are set to 0.4, 1.0 and 2.0. The Table 2 present the mean value of maximum likelihood estimator of β of the BSD along with their respective mean squared errors (MSE) and absolute bias for 1000 replicates. From Table 2, it can be observed that as the sample size increases, bias and MSE of the estimator decreases and the mean value of the estimator approaches to its original value.

6. Conclusion

In this article, we consider a new class of bimodal distributions that are symmetric in nature and named as "bimodal symmetric dstribution (BSD)". It is shown that the distribution is bimodal, symmetric and leptokurtic in nature. We have investigated several statistical properties and the parameter is estimated. Four real life data applications are considered for illustrating the usefulness of the model as compared to the existing models- bimodal symmetric Normal distribution (BND) of Elal Olivero (2010), bimodal Cauchy distribution (BCD) of Hassan and Hijazi (2010) and bimodal-symmetric Laplace distribution (BLD) of Harandi and Alamatsaz (2013). It is shown that the proposed model gives the best to all the four data sets compared to the rival modals based on certain information measures such as AIC, BIC and AICc. A brief simulation study is conducted inorder to assess the performance of the maximum likelihood estimator of the parameter of the model. Thus, it may be possible to conclude that the proposed model is more appropriate for modeling data sets in certain practical situations compared to the existing models.

Acknowledgment

The authors are very much grateful to the anonymous referees for their valuable comments on an earlier version of the paper that greatly improved the quality and presentation of the manuscript.

References

[1] Azzalini, A. and Bowman, A. W. (1990). A look at some data on the old faithful geyser. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 39(3):357-365.

[2] Behboodian, J. (1970). On the modes of a mixture of two normal distributions. *Technometrics*, 12(1):131-139.

[3] Benderly, J. and Zwick, B. (1985). Inflation, real balances, output, and real stock returns. *The American Economic Review*, 75(5):1115-1123.

[4] Chosh, P. (1978). A characterization of a bimodal distribution. *Communications in Statistics-Theory and Methods*,7(5):45-477.

[5] Dierickx, D., Basu, B., Vleugels, J., and Van der Biest, O. (2000). Statistical extreme value modeling of particle size distributions: experimental grain size distribution type estimation and parameterization of sintered zirconia. *Materials characterization*, 45(1):61-70.

[6] Eisenberger, I. (1964). Genesis of bimodal distributions. *Technometrics*, 6(4):357-363.

[7] Elal-Olivero, D. (2010). Alpha-skew-normal distribution. *Proyecciones (Antofagasta)*, 29(3):224-240.

[8] Ely, J., Fudenberg, H., Muirhead, R., LaMarche, M., Krone, C., Buscher, D., and Stern, E. (1999). Urine mercury in micromercurialism: bimodal distribution and diagnostic implications. *Bulletin of environmental contamination and toxicology*, 63:553-559.

[9] Ertel, A. (2010). Article commentary: Bimodal gene expression and biomarker discovery. *Cancer Informatics*, 9:11-14.

[10] Harandi, S. S. and Alamatsaz, M. (2013). Alpha skew laplace distribution. *Statistics and Probability Letters*,83(3):774-782.

[11] Hassan, M. Y. and Hijazi, R. H. (2010). A bimodal exponential power distribution. *Pakistan Journal Statistics*, 26(2):379-396.

[12] McLachlan, G. J., Lee, S. X., and Rathnayake, S. I. (2019). Finite mixture models. *Annual review of statistics and its application*, 6(1):355-378.

[13] McLachlan, G. J. and Peel, D. (2000). Finite mixture models. John Wiley and Sons.

[14] Rao, K., Narayana, J., and Papayya, S. V. (1988). A bimodal distribution. *Bulletin of the Calcutta Mathematical Society*, 80(4):238-240.

[15] Sarma, P., Rao, K. S., and Rao, R. P. (1990). On a family of bimodal distributions. *Sankhya: The Indian Journal of Statistics, Series B*, 52(3):287-292.

[16] Wang, J., Wen, S., Symmans, W. F., Pusztai, L., and Coombes, K. R. (2009). The bimodality index: a criterion for discovering and ranking bimodal signatures from cancer gene expression profiling data. *Cancer informatics*, 7:199-216.

[17] Weber, N. A. (1946). Dimorphism in the african oecophylla worker and an anomaly (hym.: Formicidae). *Annals of the Entomological Society of America*, 39(1):7-10.

[18] Zhang, C., Mapes, B. E., and Soden, B. J. (2003). Bimodality in tropical water vapour. *Quarterly Journal of the Royal Meteorological Society: A journal of the atmospheric sciences, applied meteorology and physical oceanography*, 129(594): 2847-2866.